Corrigé colle d'informatique n°12.

Exercice n°1

```
> with(linalg):
```

`Warning, the protected names norm and trace have been redefined and unprotected`

```
> A := evalm(matrix(5,5,1) + 1);
```

$$A := \begin{bmatrix} 2 & 1 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 & 1 \\ 1 & 1 & 2 & 1 & 1 \\ 1 & 1 & 1 & 2 & 1 \\ 1 & 1 & 1 & 1 & 2 \end{bmatrix}$$

```
> jordan(A,'P');
```

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 6 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

```
> An := n -> evalm(P &* diag(1,6^n,1,1,1) &* (1/P));
```

$$An := n \to \mathrm{evalm}(\text{‘}\&*\text{‘}(\text{‘}\&*\text{‘}(P,\ \mathrm{diag}(1,\ 6^n,\ 1,\ 1,\ 1)),\ \frac{1}{P}))$$

```
> An(n);
```

$$\begin{bmatrix} \frac{4}{5}+\frac{1}{5}6^n & -\frac{1}{5}+\frac{1}{5}6^n & -\frac{1}{5}+\frac{1}{5}6^n & -\frac{1}{5}+\frac{1}{5}6^n & -\frac{1}{5}+\frac{1}{5}6^n \\ -\frac{1}{5}+\frac{1}{5}6^n & \frac{4}{5}+\frac{1}{5}6^n & -\frac{1}{5}+\frac{1}{5}6^n & -\frac{1}{5}+\frac{1}{5}6^n & -\frac{1}{5}+\frac{1}{5}6^n \\ -\frac{1}{5}+\frac{1}{5}6^n & -\frac{1}{5}+\frac{1}{5}6^n & \frac{4}{5}+\frac{1}{5}6^n & -\frac{1}{5}+\frac{1}{5}6^n & -\frac{1}{5}+\frac{1}{5}6^n \\ -\frac{1}{5}+\frac{1}{5}6^n & -\frac{1}{5}+\frac{1}{5}6^n & -\frac{1}{5}+\frac{1}{5}6^n & \frac{4}{5}+\frac{1}{5}6^n & -\frac{1}{5}+\frac{1}{5}6^n \\ -\frac{1}{5}+\frac{1}{5}6^n & -\frac{1}{5}+\frac{1}{5}6^n & -\frac{1}{5}+\frac{1}{5}6^n & -\frac{1}{5}+\frac{1}{5}6^n & \frac{4}{5}+\frac{1}{5}6^n \end{bmatrix}$$

```
> An(1);
```

$$\begin{bmatrix} 2 & 1 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 & 1 \\ 1 & 1 & 2 & 1 & 1 \\ 1 & 1 & 1 & 2 & 1 \\ 1 & 1 & 1 & 1 & 2 \end{bmatrix}$$

Exercice n°2

```
> M := matrix(4,4,1): M[2,2] := 0: M[3,2] := 0: M[2,3] := 0: M[3,3] :=
0: evalm(M);
```

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

```
> eigenvects(M);
```

$$[1+\sqrt{5},\ 1,\ \{\left[\frac{1}{2}+\frac{1}{2}\sqrt{5},\ 1,\ 1,\ \frac{1}{2}+\frac{1}{2}\sqrt{5}\right]\}],\ [1-\sqrt{5},\ 1,\ \{\left[\frac{1}{2}-\frac{1}{2}\sqrt{5},\ 1,\ 1,\ \frac{1}{2}-\frac{1}{2}\sqrt{5}\right]\}],$$
$$[0,\ 2,\ \{[-1,\ 0,\ 0,\ 1],\ [0,\ -1,\ 1,\ 0]\}]$$

Les valeurs propres de M sont 0, $1+\sqrt{5}$ et $1-\sqrt{5}$ et les sous-espaces propres associés sont respectivement :

$$\text{Vect}\left([0,1,-1,0],[-1,0,0,1]\right),\ \text{Vect}\left(\left[\frac{1}{2}+\frac{1}{2}*\sqrt{5},1,1,\frac{1}{2}+\frac{1}{2}*\sqrt{5}\right]\right)\ \text{et Vect}\left(\left[\frac{1}{2}-\frac{1}{2}*\sqrt{5},1,1,\frac{1}{2}-\frac{1}{2}*\sqrt{5}\right]\right)$$

```
> jordan(M,'P');
```

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1-\sqrt{5} & 0 & 0 \\ 0 & 0 & 1+\sqrt{5} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

```
> A := simplify(evalm(P &* matrix(4,4,(i,j) -> if i=2 and j=2 then 1
else 0 fi) &* (1/P)));
```

$$A := \begin{bmatrix} \%1 & -\frac{1}{10}\sqrt{5} & -\frac{1}{10}\sqrt{5} & \%1 \\ -\frac{1}{10}\sqrt{5} & \%2 & \%2 & -\frac{1}{10}\sqrt{5} \\ -\frac{1}{10}\sqrt{5} & \%2 & \%2 & -\frac{1}{10}\sqrt{5} \\ \%1 & -\frac{1}{10}\sqrt{5} & -\frac{1}{10}\sqrt{5} & \%1 \end{bmatrix}$$

$$\%1 := -\frac{1}{20}\sqrt{5}+\frac{1}{4}$$

$$\%2 := \frac{1}{20}\sqrt{5}+\frac{1}{4}$$

```
> B := simplify(evalm(P &* matrix(4,4,(i,j) -> if i=3 and j=3 then 1
else 0 fi) &* (1/P)));
```

$$B := \begin{bmatrix} \%1 & \frac{1}{10}\sqrt{5} & \frac{1}{10}\sqrt{5} & \%1 \\ \frac{1}{10}\sqrt{5} & \%2 & \%2 & \frac{1}{10}\sqrt{5} \\ \frac{1}{10}\sqrt{5} & \%2 & \%2 & \frac{1}{10}\sqrt{5} \\ \%1 & \frac{1}{10}\sqrt{5} & \frac{1}{10}\sqrt{5} & \%1 \end{bmatrix}$$

$$\%1 := \frac{1}{20}\sqrt{5}+\frac{1}{4}$$

$$\%2 := -\frac{1}{20}\sqrt{5}+\frac{1}{4}$$

```
> An := n -> simplify(expand(evalm((1-sqrt(5))^n*A + (1+sqrt(5))^n*B)));
```

$$An := n \to \text{simplify}(\text{expand}(\text{evalm}((1-\sqrt{5})^n\,A+(1+\sqrt{5})^n\,B)))$$

```
> An(n);
```

$$\begin{bmatrix} \%2 & \%3 & \%3 & \%2 \\ \%3 & \%4 & \%4 & \%3 \\ \%3 & \%4 & \%4 & \%3 \\ \%2 & \%3 & \%3 & \%2 \end{bmatrix}$$

$$\%1 := (1-\sqrt{5})^n\sqrt{5}$$

$$\%2 := -\frac{1}{20}\,\%1+\frac{1}{4}(1-\sqrt{5})^n+\frac{1}{20}(1+\sqrt{5})^n\sqrt{5}+\frac{1}{4}(1+\sqrt{5})^n$$

$$\%3 := -\frac{1}{10}\,\%1+\frac{1}{10}(1+\sqrt{5})^n\sqrt{5}$$

$$\%4 := \frac{1}{20}\,\%1+\frac{1}{4}(1-\sqrt{5})^n-\frac{1}{20}(1+\sqrt{5})^n\sqrt{5}+\frac{1}{4}(1+\sqrt{5})^n$$

```
> An(2);
```

$$\begin{bmatrix} 4 & 2 & 2 & 4 \\ 2 & 2 & 2 & 2 \\ 2 & 2 & 2 & 2 \\ 4 & 2 & 2 & 4 \end{bmatrix}$$

```
> evalm(M &* M);
```

$$\begin{bmatrix} 4 & 2 & 2 & 4 \\ 2 & 2 & 2 & 2 \\ 2 & 2 & 2 & 2 \\ 4 & 2 & 2 & 4 \end{bmatrix}$$

Exercice n°3

```
> P :=  X -> X^4 + a*X^3 + sqrt(3)*X^2 + b*X;
```

$$P := X \to X^4 + a\,X^3 + \sqrt{3}\,X^2 + b\,X$$

```
> assume(a,real); assume(b,real);
> y := P(1+2*I);
```

$$y := -7 - 24\,I - (11 + 2\,I)\,a\tilde{} - (3 - 4\,I)\,\sqrt{3} + (1 + 2\,I)\,b\tilde{}$$

```
> solve({Re(y)=0, Im(y)=0},{a,b});
```

$$\{a\tilde{} = \frac{1}{2} - \frac{1}{2}\,\sqrt{3},\ b\tilde{} = \frac{25}{2} - \frac{5}{2}\,\sqrt{3}\}$$

```
> assign(%);
```

Vérification :

```
> P(X); simplify(P(1+2*I));
```

$$X^4 + (\frac{1}{2} - \frac{1}{2}\,\sqrt{3})\,X^3 + \sqrt{3}\,X^2 + (\frac{25}{2} - \frac{5}{2}\,\sqrt{3})\,X$$

$$0$$

Exercice n°4

```
> restart;
> int((a*x^5+x^4+b*x^3+c*x^2+x+d)/(x*(x^2+1)^2),x);
```

$$a\,x + d\ln(x) + \frac{1}{2}\ln(x^2+1) - \frac{1}{2}\ln(x^2+1)\,d + \frac{1}{2}\arctan(x)\,b - \frac{3}{2}\arctan(x)\,a$$
$$+ \frac{\frac{1}{4}\,((2\,a - 2\,b + 2)\,x + 2 + 2\,d - 2\,c)}{x^2+1} + \frac{1}{2}\arctan(x)$$

```
> (a*x^5+x^4+b*x^3+c*x^2+x+d)/(x*(x^2+1)^2);
```

$$\frac{a\,x^5 + x^4 + b\,x^3 + c\,x^2 + x + d}{x\,(x^2+1)^2}$$

```
> d*ln(x)+1/2*ln(x^2+1)-1/2*ln(x^2+1)*d+1/2*arctan(x)*b-3/2*arctan(x)*a
+1/2*arctan(x);
```

$$d\ln(x) + \frac{1}{2}\ln(x^2+1) - \frac{1}{2}\ln(x^2+1)\,d + \frac{1}{2}\arctan(x)\,b - \frac{3}{2}\arctan(x)\,a + \frac{1}{2}\arctan(x)$$

On prend la partie qui n'est pas une fraction rationnelle. On veut qu'elle soit nulle donc son DL doit être nul.

```
> series(%,x);
```

$$d\ln(x) + (\frac{1}{2} + \frac{1}{2}\,b - \frac{3}{2}\,a)\,x + (\frac{1}{2} - \frac{1}{2}\,d)\,x^2 + (-\frac{1}{6}\,b - \frac{1}{6} + \frac{1}{2}\,a)\,x^3 + (\frac{1}{4}\,d - \frac{1}{4})\,x^4 +$$
$$(\frac{1}{10}\,b - \frac{3}{10}\,a + \frac{1}{10})\,x^5 + \mathrm{O}(x^6)$$

Pour que le DL soit nul, il faut avoir à la fois $d = 0$ et $\frac{1}{2} - \frac{1}{2}d = 0$ ce qui est impossible. Conclusion : pas de solution.

Exercice n°5 :

```
> P :=  X -> X^4 + a*X^3 + sqrt(3)*X^2 + b*X;
```

$$P := X \to X^4 + a\,X^3 + \sqrt{3}\,X^2 + b\,X$$

```
> assume(a,real); assume(b,real);
> y := P(1+2*I);
```

$$y := -7 - 24\,I - (11 + 2\,I)\,a\tilde{} - (3 - 4\,I)\,\sqrt{3} + (1 + 2\,I)\,b\tilde{}$$

```
> solve({Re(y)=0, Im(y)=0},{a,b});
```

$$\{a\tilde{} = \frac{1}{2} - \frac{1}{2}\sqrt{3},\ b\tilde{} = \frac{25}{2} - \frac{5}{2}\sqrt{3}\}$$

```
> assign(%);
```

Vérification :

```
> P(X); simplify(P(1+2*I));
```

$$X^4 + (\frac{1}{2} - \frac{1}{2}\sqrt{3})\,X^3 + \sqrt{3}\,X^2 + (\frac{25}{2} - \frac{5}{2}\sqrt{3})\,X$$

$$0$$

Exercice n°6

```
> restart: A := n -> t^n + 1/t^n;
```

$$A := n \to t^n + \frac{1}{t^n}$$

```
> A(n) - A(1)*A(n-1)+A(n-2);
```

$$t^n + \frac{1}{t^n} - (t + \frac{1}{t})\,(t^{(n-1)} + \frac{1}{t^{(n-1)}}) + t^{(n-2)} + \frac{1}{t^{(n-2)}}$$

```
> simplify(%);
```

$$0$$

La formule précédente nous permet de démontrer par rrécurrence l'existence de P_n avec $P_0 = 2$, $P_1 = X$

est la relation de récurrence $P_n = X * P_{n-1} - P_{n-2}$.

```
> P := proc(n :: nonnegint )
if n = 0 then return(X ->2)
elif n = 1 then return(X->X)
else return(unapply(simplify(X*P(n-1)(X)-P(n-2)(X)),X)) fi;
end;
```

$P :=$ **proc**$(n{::}nonnegint)$
if $n = 0$ **then return** 2
elif $n = 1$ **then return** $X \to X$
else return unapply(simplify($X *$ P$(n-1)(X) -$ P$(n-2)(X)$), X)
end if
end proc

```
> P(10);
```

$$X \to X^{10} - 10\,X^8 + 35\,X^6 - 50\,X^4 + 25\,X^2 - 2$$

```
> for i from 0 to 10 do
Pn := P(i)(2*cos(x));
combine(Pn,trig);
od;
```

$$Pn := 2$$

$$2$$

$$Pn := 2\cos(x)$$

$$2\cos(x)$$
$$Pn := 4\cos(x)^2 - 2$$
$$2\cos(2\,x)$$
$$Pn := 8\cos(x)^3 - 6\cos(x)$$
$$2\cos(3\,x)$$
$$Pn := 16\cos(x)^4 - 16\cos(x)^2 + 2$$
$$2\cos(4\,x)$$
$$Pn := 32\cos(x)^5 - 40\cos(x)^3 + 10\cos(x)$$
$$2\cos(5\,x)$$
$$Pn := 64\cos(x)^6 - 96\cos(x)^4 + 36\cos(x)^2 - 2$$
$$2\cos(6\,x)$$
$$Pn := 128\cos(x)^7 - 224\cos(x)^5 + 112\cos(x)^3 - 14\cos(x)$$
$$2\cos(7\,x)$$
$$Pn := 256\cos(x)^8 - 512\cos(x)^6 + 320\cos(x)^4 - 64\cos(x)^2 + 2$$
$$2\cos(8\,x)$$
$$Pn := 512\cos(x)^9 - 1152\cos(x)^7 + 864\cos(x)^5 - 240\cos(x)^3 + 18\cos(x)$$
$$2\cos(9\,x)$$
$$Pn := 1024\cos(x)^{10} - 2560\cos(x)^8 + 2240\cos(x)^6 - 800\cos(x)^4 + 100\cos(x)^2 - 2$$
$$2\cos(10\,x)$$

Exercice n°7

```
> restart: with(linalg):
```

```
Warning, the protected names norm and trace have been redefined and
unprotected
```

```
> A:= matrix(3,3,[a,b,c,b,a+c,b,c,b,a]);
```

$$A := \begin{bmatrix} a & b & c \\ b & a+c & b \\ c & b & a \end{bmatrix}$$

```
> Dia := jordan(A,'P');
```

$$Dia := \begin{bmatrix} -c+a & 0 & 0 \\ 0 & c+a-\sqrt{2}\,b & 0 \\ 0 & 0 & c+a+\sqrt{2}\,b \end{bmatrix}$$

```
> evalm(P); # P est la matrice de passage dont les vecteurs colonnes
sont les vecteurs propres
```

$$\begin{bmatrix} \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ 0 & -\frac{1}{4}\sqrt{2} & \frac{1}{4}\sqrt{2} \\ \frac{-1}{2} & \frac{1}{4} & \frac{1}{4} \end{bmatrix}$$

On remarque que la matrice P ne dépend pas des coefficients a,b et c. On vérifie le résultat.

```
> simplify(evalm((1/P) &* A &* P));
```

$$\begin{bmatrix} -c+a & 0 & 0 \\ 0 & c+a-\sqrt{2}\,b & 0 \\ 0 & 0 & c+a+\sqrt{2}\,b \end{bmatrix}$$

calcul de A^n

```
> evalm(P &* diag(Dia[1,1]^n,Dia[2,2]^n,Dia[3,3]^n) &* (1/P));
```

$$\begin{bmatrix} \frac{1}{2}(-c+a)^n + \frac{1}{4}\%2 + \frac{1}{4}\%1 & \%3 & -\frac{1}{2}(-c+a)^n + \frac{1}{4}\%2 + \frac{1}{4}\%1 \\ \%3 & \frac{1}{2}\%2 + \frac{1}{2}\%1 & \%3 \\ -\frac{1}{2}(-c+a)^n + \frac{1}{4}\%2 + \frac{1}{4}\%1 & \%3 & \frac{1}{2}(-c+a)^n + \frac{1}{4}\%2 + \frac{1}{4}\%1 \end{bmatrix}$$

$\%1 := (c+a+\sqrt{2}\,b)^n$

$\%2 := (c+a-\sqrt{2}\,b)^n$

$\%3 := -\frac{1}{4}\sqrt{2}\,\%2 + \frac{1}{4}\sqrt{2}\,\%1$

```
> simplify(subs({n=1},%)); # On vérifie que ça marche pour n=1.
```

$$\begin{bmatrix} a & b & c \\ b & a+c & b \\ c & b & a \end{bmatrix}$$